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PROGRESS REPORT.

The following gentlemen have been elected members of the Society :—

(1) MR. E. Subrahmania Iyer, B.A., B.L., Vakil, District Court, Kottayam, Travancore.

(2) Mr. N. Ragunathan, M.A., Assistant Professor of Mathematics, 41, Car Street, Triplicane, MADRAS (*Concession rate*).

2. The following books have been received from the publishers :—

(1) *The School Algebra.*—(Matriculation Edition). By A. G. Cracknell, University Tutorial Press, 4/6. London, 1913.

(2) *A First Book of Practical Mathematics.*—By T. S. Usherwood and C. J. A. Trimble, MacMillan & Co., London, 1913. 1/6.

(3) *Practical Geometry and Graphics.*—By J. Harrison and G. A. Baxandall, MacMillan & Co., 6, London, 1913.

3. The proposed meeting of the Society during the next Xmas has been postponed to some future date to be notified hereafter, owing to certain unavoidable circumstances.

POONA, 30th Nov. 1913.

D. D. KAPADIA,
Hon. Joint Secretary.

600 008.

On Tetrahedral Coordinates.

By A. C. L. Wilkinson, M.A., F.R.A.S.

(*Continued from page* 130.)

§ 22. *The Sphere.*

If $f(\alpha\beta\gamma\delta) \equiv (a, b, c, d, f, g, h, u, v, w)\ (\alpha,\beta,\gamma,\delta,)^2=0$ represent a sphere and $(\alpha',\beta',\gamma',\delta')$ is its centre, then by considering its intersection with

$$\frac{\alpha-\alpha'}{l}=\frac{\beta-\beta'}{m}=\frac{\gamma-\gamma'}{n}=\frac{\delta-\delta'}{p}=\frac{r}{k}, \text{ where } k^2=-\Sigma lm\,AB^2,$$

we have

$$f\left(\alpha'+l\frac{k}{r},\beta'+m\frac{k}{r},\ \gamma'+n\frac{k}{r},\ \delta'+p\frac{k}{r}\right)=0\,;$$

or

$$f(\alpha',\beta'.\gamma',\delta')+\frac{r}{k}\Sigma l\frac{\partial f}{\partial\alpha'}+\frac{r^2}{k^2}f(l, m, n, p)=0.$$

The roots of this equation in r must be equal and opposite for all values of $l+m+n+p=0$.

The centre is therefore given by

$$\frac{\partial f}{\partial\alpha'}=\frac{\partial f}{\partial\beta'}=\frac{\partial f}{\partial\gamma'}=\frac{\partial f}{\partial\delta'}.$$

Further $\dfrac{f(l, m, n, p)}{k^2}$ must be independent of l, m, n, p; substituting for l^2, m^2, n^2, p^2 the values $-l(m+n+p)$, etc., we obtain the conditions for a sphere, *viz*,

$$\frac{2f-b-c}{BC^2}=\frac{2g-c-a}{CA^2}=\frac{2h-a-b}{AB^2}=\frac{2u-a-d}{AD^2}$$
$$=\frac{2v-b-d}{BD^2}=\frac{2w-c-d}{CD^2}=-\frac{f(l, m, n, p)}{k^2}=\frac{1}{\lambda}.$$

The radius of the sphere is given by

$$R^2=\lambda f(\alpha',\beta',\gamma',\delta').$$

Example: 1. The sphere circum scribing the tetrahedron is

$$\Sigma AB^2\alpha\beta=0.$$

The centre is given by

$$\begin{aligned}
AB^2\beta'+AC^2\gamma'+AD^2\delta' &=\lambda,\\
AB^2\alpha'\qquad\quad +BC^2\gamma'+BD^2\delta' &=\lambda,\\
AC^2\alpha'+BC^2\beta'\qquad\quad +CD^2\delta' &=\lambda,\\
AD^2\alpha'+BD^2\beta'+CD^2\gamma'\qquad\quad &=\lambda,\\
\alpha'+\beta'+\gamma'+\delta' &=1,
\end{aligned}$$

whence α' is proportional to k_1 of § 11. Further $R^2=f(\alpha',\beta',\gamma',\delta^2)=\frac{1}{2}\lambda$, by use of the relations giving the centre.

Thus R^2 is given by

$$\begin{vmatrix} 0, & AB^2, & AC^2, & AD^2, & 2R^2 \\ AB^2, & 0, & BC^2, & BD^2, & 2R^2 \\ AC^2, & BC^2, & 0, & CD^2, & 2R^2 \\ AD^2, & BD^2, & CD^2, & 0, & 2R^2 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0.$$

This agrees with the well-known expressions.

Ex. 2. Consider the tetrahedron such that the six middle points of its edges lie on a sphere.

The point (1,1,00) gives the condition $a+b+2f=0$, so that

$$\frac{a+b}{AB^2}=\frac{a+c}{AC^2}=\text{etc.},$$

and the conditions necessary and sufficient are

$$AB^2+CD^2=AC^2+BD^2=AD^2+BC^2.$$

The sphere is $\Sigma a\,\alpha^2-\Sigma(a+b)\alpha\beta=0$.

Where this sphere again meets AB or $\gamma=0$, $\delta=0$, we have

$$a\alpha^2+b\beta^2-(a+b)\alpha\beta=0\,;$$

or $\alpha=\beta$ and $a\alpha=b\beta$ and the latter point is the foot of the perpendicular from C or D on AB. For, if X is this point

$$\frac{\alpha}{\beta}=\frac{BX}{XA}=\frac{BC\cos ABC}{AC\cos BAC}=\frac{AB^2+BC^2-CA^2}{AB^2+AC^2-BC^2}$$

$$=\frac{a+b+b+c-(c+a)}{a+b+c+a-(b+c)}=\frac{b}{a}.$$

This again is a well-known elementary theorem.

§ 23. *To determine the principal axes of the general conicoid.*

We shall write the conicoid referred to rectangular axes

$$a'x^2+b'y^2+c'z^2+2h'xy+2g'xz+2f'yz+2u'x+2v'y+2w'z+d'=0;$$

and referred to the tetrahedron

$$a\alpha^2+b\beta^2+c\gamma^2+d\delta^2+2f\beta\gamma+2g\alpha\gamma+2h\alpha\beta+2u'\alpha\delta+2v'\beta\delta+2w\gamma\delta=0.$$

The linear transformation
$$\begin{aligned} x&=\alpha x_1+\beta x_2+\gamma x_3+\delta x_4\\ y&=\alpha y_1+\beta y_2+\gamma y_3+\delta y_4\\ z&=\alpha z_1+\beta z_2+\gamma z_3+\delta z_4\\ 1&=\alpha+\beta+\gamma+\delta\end{aligned}$$

has for its determinant 6 V, where V is the volume of the tetrahedron.

Thus
$$36\,V^2\begin{vmatrix} a', & h', & g', & u' \\ h', & b', & f', & v' \\ g', & f', & c', & w' \\ u', & v', & w', & d' \end{vmatrix} = \begin{vmatrix} a, & h, & g, & u \\ h, & b, & f, & v \\ g, & f, & c, & w \\ u, & v, & w, & d \end{vmatrix}.$$

Changing d' to $d'+\lambda$, we change $a, g\ldots\ldots$to $a+\lambda$, $h+\lambda$, $g+\lambda,\ldots$ and by equating the coefficients of λ, we obtain

$$36\,V^2\begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = -\begin{vmatrix} a & h & g & u & 1 \\ h & b & f & v & 1 \\ g & f & c & w & 1 \\ u & v & w & d & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

The determinant giving the squares of the reciprocals of the principal axes is

$$\begin{vmatrix} a'-\lambda, & h', & g' \\ h', & b'-\lambda, & f' \\ g', & f', & c'-\lambda \end{vmatrix} = 0, \text{ where } \lambda = -\frac{1}{r^2}\begin{vmatrix} a' & h' & g' & u' \\ h' & b' & f' & v' \\ g' & f' & c' & w' \\ u' & v' & w' & d' \end{vmatrix} \div \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix}$$

which is the discriminant of $a'x^2+\ldots+2h'xy+\ldots+2u'x+\ldots+d'-\lambda(x^2+y^2+r^2)$. The transformed conicoid has coefficients

$$a-\lambda(x_1^2+y_1^2+z_1^2)\ldots, h-\lambda(x_1x_2+y_1y_2+z_1z_2)\ldots,$$

Thus the axes are given by

$$\begin{vmatrix} a-\lambda a_1, & h-\lambda h_1, & g-\lambda g_1, & u-\lambda u_1, & 1 \\ h-\lambda h_1, & b-\lambda b_1, & f-\lambda f_1, & v-\lambda v_1, & 1 \\ g-\lambda g_1, & f-\lambda f_1, & c-\lambda c_1, & w-\lambda w_1, & 1 \\ u-\lambda u_1, & v-\lambda v_1, & w-\lambda w_1, & d-\lambda d_1, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}$$

where $a_1\ldots, h_1\ldots$ are written for $x_1^2+y_1^2+z_1^2, \ldots, x_1x_2+y_1y_2+z_1z_2, \ldots\ldots$

By bordering the determinant and by some simple reductions, we obtain for the determinant giving the magnitudes of the squares of the reciprocals of the axes

$$\begin{vmatrix} a, & h, & g, & u, & 1, & x_1 & y_1 & z_1 \\ h. & b, & f, & v, & 1, & x_2, & y_2, & z_2 \\ g, & f, & c, & w, & 1, & x_3, & y_3, & z_3 \\ u, & v, & w, & d, & 1, & x_4, & y_4, & z_4 \\ 1, & 1, & 1, & 1, & 0, & 0, & 0, & 0 \\ x_1, & x_2, & x_3, & x_4, & 0, & 1/\lambda, & 0, & 0 \\ y_1, & y_2, & y_3, & y_4, & 0, & 0, & 1/\lambda, & 0 \\ z_1, & z_2, & z_3, & z_4, & 0, & 0, & 0, & 1/\lambda \end{vmatrix}$$

This may be expanded without difficulty in terms of the lengths of the edges of the tetrahedron.

It will be say $A/\lambda^3+B/\lambda^2+C/\lambda+D=0$:

Here A is the determinant of the first 25 elements and is of three dimensions in the letters, $a \ldots, h \ldots$; B is of two dimensions in the letters; C of one dimension and D is 36 V^2, as is easily seen by putting $1/\lambda$; everwhere zero.

C consists of typical terms of which the calculation of the coefficients of a, and h will enable us to write down the value of C by symmetry.

The coefficient of a is
$$\begin{vmatrix} 0, & 0, & 0, & 1, & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 & z_3 \\ 0 & 0 & 0 & 1 & x_4 & y_4 & z_4 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ x_2 & x_3 & x_4 & 0 & 1 & 0 & 0 \\ y_2 & y_3 & y_4 & 0 & 0 & 1 & 0 \\ z_2 & z_3 & z_4 & 0 & 0 & 0 & 1 \end{vmatrix}$$

since, obviously, all the other letters can be replaced by zeros.

Expand this by Lagranges rule taking sets of determinants of 9 elements from the first three columns ; we find

$$-\sum\begin{vmatrix} 1 & 1 & 1 \\ x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \end{vmatrix}^2 = -4\triangle\text{BCD}^2.$$

For the co-efficient of h, we have to note that h occurs twice ; in each case its coefficient is

$$-\begin{vmatrix} 0 & 0 & 0 & 1 & x_2 & y_2 & z_2 \\ 0 & 0 & 0 & 1 & x_3 & y_3 & z_3 \\ 0 & 0 & 0 & 1 & x_4 & y_4 & z_4 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ x_1 & x_3 & x_4 & 0 & 1 & 0 & 0 \\ y_1 & y_3 & y_4 & 0 & 0 & 1 & 0 \\ z_1 & z_3 & z_4 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Reduce all the elements of the determinant formed by the first three columns and the last three rows to zero by use of the determinant formed by the last three rows and columns, and we get

$$-\begin{vmatrix} x_1x_2+y_1y_2+z_1z_2, & x_2x_3+y_2y_3+z_2z_3, & x_2x_4+y_2y_4+z_2z_4, & 1 \\ x_1x_3+y_1y_3+z_1z_3 & x_3^2+y_3^2+z_3^2 \quad , & x_3x_4+y_3y_4+z_3z_4, & 1 \\ x_1x_4+y_1y_4+z_1z_4 & x_3x_4+y_3y_4+z_3z_4, & x_4^2+y_4^2+z_4^2 \quad , & 1 \\ 1 \quad , & 1 \quad , & 1 \quad , & 0 \end{vmatrix}$$

By taking the origin at (x_4, y_4, z_4) we may write this

$$(x_1x_2+y_1y_2+z_1z_2)(x^2{}_3+y^2{}_3+z^2{}_3)-(x_2x_3+y_2y_3+z_2z_3)(x_1x_3+y_1y_3+z_1z_3).$$

Now $x_3^2+y_3^2+z_3^2=\text{DC}^2$

$$x_1x_2+y_1y_2+z_1z_2=\tfrac{1}{2}\left\{x_1^2+y_1^2+z_1^2+x_2^2+y_2^2+z_2^2-(x_1-x_2)^2-(y_1-y_2)^2-(z_1-z_2)^2\right\}$$

$$=\tfrac{1}{2}(\text{DA}^2+\text{DB}^2-\text{AB}^2)\,;$$

Hence the coefficient of h reduces to

$$\tfrac{1}{2}\,(\text{DA}^2\text{BC}^2+\text{DB}^2\text{AC}^2-\text{DC}^2\text{AB}^2)-\tfrac{1}{4}(\text{DC}^2-\text{DA}^2-\text{CA}^2)(\text{DC}^2-\text{DB}^2-\text{CB}^2)=\tfrac{1}{4}f(\text{CD}),$$

which is symmetrical as far as the combination $\alpha\beta$ or AB on which h depends. This is to be doubled in the final result.

We have now to consider the expression B :

B consists of all terms that occur, of two dimensions in the lettesr $a_1 \ldots h_1 \ldots$ The types of terms to be evaluated reduce to ab, h^2, ah,

af, hg, hw corresponding to factors of the forms $\alpha^2\times\beta^2$, $\alpha\beta\times\alpha\beta$ $\alpha^2\times\alpha\beta$, $\alpha^2\times\beta\gamma$, $\alpha\beta\times\alpha\gamma$, $\alpha\beta\times\gamma\delta$. We notice that $ab-h^2$ occurs only in this combination, ah does not occur in the determinant, af and hg occur each twice and hw occurs four times.

The coefficient of $ab-h^2$ is

$$\begin{vmatrix} 0 & 0 & 1 & x_3 & y_3 & z_3 \\ 0 & 0 & 1 & x_4 & y_4 & z_4 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ x_3 & x_4 & 0 & 1 & 0 & 0 \\ y_3 & y_4 & 0 & 0 & 1 & 0 \\ z_3 & z_4 & 0 & 0 & 0 & 1 \end{vmatrix} = -\begin{vmatrix} \Sigma x_3^2 & \Sigma x_3x_4 & 1 \\ \Sigma x_3x_4 & \Sigma x^2_4 & 1 \\ 1 & 1 & 0 \end{vmatrix} = \mathrm{CD}^2$$

since x_4, y_4, z_4 can all be written zero.

The coefficient of $(af-hg)$ is

$$-\begin{vmatrix} 0 & 0 & 1 & x_3 & y_3 & z_3 \\ 0 & 0 & 1 & x_4 & y_4 & z_4 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ x_2 & x_4 & 0 & 1 & 0 & 0 \\ y_2 & y_4 & 0 & 0 & 1 & 0 \\ z_2 & z_4 & 0 & 0 & 0 & 1 \end{vmatrix}$$

which reduces, on writing $x_4y_4z_4$ all zero, to $-(x_2x_3+y_2y_3+z_2z_3)$, which is equal to $\frac{1}{2}(\mathrm{CB}^2-\mathrm{DB}^2-\mathrm{DC}^2)$. This coefficient its to be doubled. The terms hw are of two types in the determinant; we thus have for the coefficient

$$\begin{vmatrix} 0 & 0 & 1 & x_2 & y_2 & z_2 \\ 0 & 0 & 1 & x_4 & y_4 & z_4 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ x_1 & x_3 & 0 & 1 & 0 & 0 \\ y_1 & y_3 & 0 & 0 & 1 & 0 \\ z_1 & z_3 & 0 & 0 & 0 & 1 \end{vmatrix} + \begin{vmatrix} 0 & 0 & 1 & x_2 & y_2 & z_2 \\ 0 & 0 & 1 & x_3 & y_3 & z_3 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ x_1 & x_4 & 0 & 1 & 0 & 0 \\ y_1 & y_4 & 0 & 0 & 1 & 0 \\ z_1 & z_4 & 0 & 0 & 0 & 1 \end{vmatrix}$$

Putting $x_4y_4z_4$ zero, we obtain

$$-\Sigma x_2x_3+\Sigma x_1x_2-\Sigma x_1x_3+\Sigma x_1x_2$$

which is the same as

$$-\tfrac{1}{2}\{DB^2+DC^2-BC^2+DA^2+DC^2-AC^2\}+AB^2-DA^2-DB^2,$$

or

$$-\tfrac{1}{2}\{2\,AB^2+2\,DC^2-DA^2-DB^2-CA^2-CB^2\}.$$

Thus finally, we have the equation giving the squares of the reciprocals of the axes

$$36\,V^2\lambda^3+C\lambda^2+B\lambda+A=0,$$

where

$$C=\{\Sigma h[DA^2.BC^2+DB^2.AC^2-DC^2.AB^2 -\tfrac{1}{2}(DC^2-DA^2-CA^2)(DC^2-DB^2-CB^2)] -4\Sigma a(\triangle BCD)^2.\}$$

$$B=\{\Sigma(ab-h^2)CD^2+\Sigma(af-hg)(CB^2-DB^2-DC^2). -\Sigma hw(2AB^2+2CD^2-DA^2-DB^2-CA^2-CB^2)\}.$$

$$A=\begin{vmatrix} a & h & g & u & 1 \\ h & b & f & v & 1 \\ g & f & c & w & 1 \\ u & v & w & d & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

and

$$\lambda=\frac{1}{r^2}\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix}\div A.$$

This equation could be obtained as in Ferrer's *Trilinear Coordinates*. The equation in fact is

$$\begin{vmatrix} a, & h+\tfrac{1}{2}\lambda AB^2, & g+\tfrac{1}{2}\lambda AC^2, & u+\tfrac{1}{2}\lambda AD^2, & 1 \\ h+\tfrac{1}{2}\lambda AB^2, & b, & f+\tfrac{1}{2}\lambda BC^2, & v+\tfrac{1}{2}\lambda BD^2, & 1 \\ g+\tfrac{1}{2}\lambda AC^2, & f+\tfrac{1}{2}\lambda BC^2, & c, & w+\tfrac{1}{2}\lambda CD^2, & 1 \\ u+\tfrac{1}{2}\lambda AD^2, & v+\tfrac{1}{2}\lambda BD^2, & w+\tfrac{1}{2}\lambda CD^2, & d, & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}=0$$

§ 24. As an application, take the hyperboloid found in § 14 (1). Writing L, M, N for $f(AD)\,f(BC)$, $f(AC)\,f(BD)$, $f(AB)\,f(CD)$, consider the determinant

$$\begin{vmatrix} 0, & f(AB)(L-M), & f(AC)(N-L), & f(AD)(M-N), & 1 \\ f(AB)(L-M), & 0, & f(BC)M-N), & f(BD)(N-L), & 1 \\ f(AC)(N-L), & f(BC)(M-N), & 0, & f(CD)(L-M), & 1 \\ f(AD)(M-N), & f(BD)(N-L), & f(CD)(L-M) & 0 & 1 \\ 1, & 1, & 1, & 1, & 0 \end{vmatrix}$$

We note that L−M is a factor, for ; putting L=M, it becomes

$$(L-N)^3 \begin{vmatrix} 0 & 0 & -f(AC), & f(AD), & 1 \\ 0 & 0 & f(BC), & -f(BD), & 1 \\ -f(AC), & f(BC) & 0 & 0, & 1 \\ f(AD), & -f(BD) & 0 & 0, & 1 \\ 1 & 1 & 1 & 1 & 0 \end{vmatrix}$$

and the determinant here is zero, since, it has $f(AC)\,f(BC)-f(AD)\,f(BC)$ $(L-M)$ as a factor.

Thus $(L-M)(M-N)(N-L)$ is a factor and from symmetry we assume the determinant equal to

$$(L-M)(M-N)(N-L)[lS_1+mS_2+nS_3+pS_4+qS_5+rS_6],$$

where l, m, n, p, q, r are some constants, and

$$S_1=\Sigma f(BC)f(CD)f(BD), \; S_2=\Sigma f(AB)f(AC)f(AD),$$

$$S_3=\Sigma f(AD)f(BC).\, \{ f(AB)+f(AC)+f(BD)+f(CD) \}$$

$$S_4=\Sigma f(AD)f(BC) \{ f(AD)+f(BC) \}$$

$$S_5=f(AB)^2 \} f(AC)+f(AD)+f(BC)+f(BD) \}$$

$$S_6=\Sigma \{ f(AB)^3 \},$$

Write $f(AD)=0=f(BC)$, and the determinant reduces very considerably and we find

$$n=-2,\; p=0,\; q=0,\; r=0.$$

Again, keep only the highest power of $f(AB)$ and so find

$$l=-2, m=0.$$

Thus by § **11** (6) the determinant is equal to

$$-2\Delta^2(L-M)(M-N)(N-L).$$

Again, consider the determinant

$$\begin{vmatrix} 0, & f(AB)(L-M), & f(AC)(N-L), & f(AD)(M-N) \\ f(AB)(L-M), & 0 \quad , & f(BC)(M-N), & f(BD)(N-L) \\ f(AC)(N-L), & f(BC)(M-N), & 0 \quad , & f(CD)(L-M) \\ f(AD)(M-N), & f(BD)(N-L), & f(CD)(L-M), & 0 \end{vmatrix}.$$

$$=\Sigma f(AB)^2 f(CD)^2 (L-M)^4 - 2\Sigma f(AB)\cdot f(CD)\cdot f(AC).\, f(BD) \\ (L-M)^2(N-L)^2.$$

This is divisible by $(L-M)^2$ provided that

$(N-L)^2 f(AC) f(BD) - f(AD) f(BC)(M-N)^2$ is divisible by $L-M$, which is the case, since it vanishes for $L=M$.

Hence our second determinant is equal to $(L-M)^2\,(M-N)^2\,(N-L)^2$ and λ of § 23 is equal to

$$-\frac{1}{r^2}.\frac{(L-M)\ (M-N)\ (N-L)}{2\cdot(288\ V^2)^2}$$

Thus
$$r_1^2 r_2^2 r_3^2 = -\frac{(L-M)^2\ (M-N)^2\ (N-L)^2}{8\cdot(288V^2)^6.\ 2(288V^2)^2}.36V^2,$$

and by use of § 11 (1), $r_1 r_2 r_3$ is equal to$_3$

$$\frac{(AB^2+CD^2-AC^2-BD^2)(AB^2+CD^2-AD^2-BC^2)(AD^2+BC^2-AC^2-BD^2)}{192V}$$

Mathematical Teaching in the Secondary Schools.*

The first thing that must strike any observer is that we are passing through a period of great change in all that pertains to Secondary Education. A subject like Descriptive Geometry, for example, can hardly be said to occupy any definite position ; so rapidly changing are the views concerning its importance, its significance, and the schools in which it should be taught.

Austria makes much of it in the Realschule, less of it in the Realgymnasium, and almost nothing of it in the Gymnasium. This is what we might expect, but what descriptive geometry will do in the Reformrealgymnasium it is rather early to say, since this type of school is only beginning to appear in that country. In England there is seen the same spirit of unrest, and the training of boys in special lines is having its influence in the general fields as well. The teaching of boys who are going into the army, engineering, or surveying, tends to assume a more intuitional and experimental character than that of other boys, and the army requirements during the last *ten* years have no doubt had a powerful reaction on the character of the teaching throughout the English public schools. On the other hand, as regards the teaching of boys other than specialists, probably the prevalent view in England is that the proper field for methods of intuition and experiment is in the middle and lower classes rather than the upper. These methods are viewed with suspicion by many masters who are concerned with the more able mathematical boys, and in many cases practical methods learnt by the boys in the lower classes are allowed to rest in the upper. Many teachers consent to the postponement of abstract methods during the earlier teaching, only on the understanding that the abstract character of the higher teaching shall be preserved.

In the United States the spirit of unrest in all educational matters is also manifest, and in particular with respect to this whole question of intuition and experiment in mathematics. The question has been agitated for the past ten years, and many kinds of experiments have been tried, varying from an extreme laboratory method with a *minimum* of mathematics to the most abstract kind of work in which intuition and experiment play almost no part. It seems quite impossible to report as to any fixed policy or general consensus of opinion upon the subject.

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* Extracted from the Report presented by Prof. D. E. Smith to the *Fifth International Congress of Mathematicians*, Cambridge, 1912.

We shall now take up *seriatim* the various topics set forth in the *questionnaire* that was sent out by Dr. Lietzmann last winter. Since it will not be profitable to present the report in full, a brief summary of the results will now be given.

In the work of measuring and estimating a more practical form of mensuration seems to be developing, especially in Austria, Germany, and Switzerland. England, France, and the United States seem to have given the matter less attention, or at least to have secured less definite results. An elementary trigonometry is more commonly found at an earlier period in the first three countries, thus allowing for outdoor work with simple instruments at an earlier stage.

In the matter of geometric drawing and graphic representation of solids, the various countries seem to be in a transition stage between the period in which this was considered part of the duties of the art teacher, and that in which it is to be taken over by the department of mathematics. The tendency is general to consider this work as part of mathematics. The nature of the work is, however, not at all settled; even the term "descriptive geometry" has no well-defined meaning. It may be said in a general way that the subject is taught in schools of the Oberrealschule type, but not, in those of the Gymnasium type, and that the tendency is to increase the work both in amount and quality in the former.

Graphic methods of representing functions have become universal in the last generation. From the idea of a line representing an equation the tendency is at present to that of a graphic representation of a function. Just how much the pupil is acquiring of the function concept seems often to be questioned, and the whole subject is in the experimental stage at present. Of the value of squared millimetre paper there is no question anywhere, but it seems equally true that its use has been abused by the over extensive treatment of equations and by its application to proving the obvious.

The contracted methods of computation that were prominently advocated fifty years ago do not seem to have advanced materially, owing to the feeling that they are not really practical. On the other hand the use of logarithms seems on the increase, and the slide rule has come into great favour in the technical schools where approximate calculation is prominent. Graphic methods of computation and of the approximation of roots of numerical higher equations have found but little place in the schools of the type under investigation. It is probably too early to say what their success will be in this kind of school

or when, if ever, any large body of teachers will be found capable of treating the subject.

In general, it may be said that more progress towards the recognition of the *role* of intuition and experiment in secondary mathematics seems to have been made of late in Austria, Germany, and Switzerland, than in England, France, and the United States. A comparison of the work cone in these several countries, particularly in the way of applying mathematics seriously to the problems of life, and visualizing the work in mensuration, will be one of the valuable results of the present International movement.

The great questions of all, however, relate to the nature of geometry and to the treatment of the function concept. Other questions are also important, but here is the dominant issue that must be met in the next few years. The first of these questions is this: How much of the Geometry of the Secondary Schools shall be inductive, and how much deductive? Few are ready to assert to-day that it is best to begin Geometry with a study of Euclid or Legendre. There must be a preparatory stage, and this must be characterized by intuition and experiment. But how much time shall be assigned to this stage; and exactly what ground shall be covered; and what is more important, to what extent shall intuition replace deduction of the Euclid type? † Must we have two or even three years of *Anschauungslehre*, as some have advocated, or is a year or a half-year enough? How much must the rigorist in geometry be compelled to concede to the demands of the non-mathematical mind? Was Newton right when he expressed the opinion that all intuitive work is pretty, but that is not geometry? Must the real geometry of the past go by the board, as went the mediaeval logic, or will its position be strengthened by this propaedeutic work? Is the value of the Euclidean type of geometry such as to save it from destruction, or is it to be so diluted by this consideration of pictures, of models, and of simple mensuration as to be unrecognizable? Is the overpowering force of to-day, the force of modern industry, the cause of the growth of intuition and experiment in geometry? And, if so, what will be the overpowering force of to-morrow, the force of special considerations, demand? These are questions that we hear about us, but these are questions which we are not yet able to answer scientifically. In general it may be said that it is the plan of the Teutonic countries to mix the intuitional and the deductive work from the outset, while in France, and now in England, the

† Readers are referred to the recent works by Truetline and Fimerding, published by Teubner.

plan is to let an inductive cycle precede a deductive one. The United States is only beginning to talk about the question whatever tendency there is towards the Anglo-French plan. Now is either of these plans better than the other; and can this be proved? Or is the question one of racial habit? Would the German plan succeed in the United States, or is the French plan better adapted to such a conglomeration of races? Would the English plan of arranging the work in three stages, with the intuition and experiment the initial one, be better for Germany than the totally different scheme that characterizes most of the recent text-books of that country? Many people look upon the German plan as unscientific, attributing its apparent success to the excellently trained teachers who carry it out; while others look upon the cycle arrangement as ultrascientific. Which is right? Or is neither right?

The second important question relates to the treatment of the function concept. Here the *role* of intuition, in the first steps, is more clearly defined, since we have no welltried body of knowledge to be set aside. The chief argument for the elaboration of the function concept seems to be that the calculus has already found a place in the schools under our consideration, and if it is to hold its place and continue to grow in strength, we must cease to impose it merely from above—we must prepare for it from below. The notions of limit, invariability, rate, function, and graph must be so gradually introduced and must become so clearly understood that when the calculus is reached they will be met as we meet familiar friends. How to do this economically is one of the problems relating to intuitional mathematics. It is one of the interesting facts of present education that teachers are demanding the elimination of the incommensurable quantity from elementary demonstrative geometry, only to find themselves face to face with a demand for the study of limits, functions, and rate of change. The movement in favour of the elaboration of the function concept, however, is too recent to judge of its permanence in secondary education. Starting in France within the last twenty years, and vigorously advocated in Germany within the last decade, it has much to commend it if reasonably treated.

REVIEWS.

Elementary Trigonometry : by W. E. Paterson, M.A., B.Sc.

(Oxford, Henry Frowde, 1911.)

This book is written on very fresh lines, and gives an adequate treatment of Trigonometry up to and inclusive of the Solution of Triangles. A supplement of 28 pages gives Logarithmic and Trigonometric Tables, the arrangement being an improvement on the usual plan of present-day text-books.

The writer insists on the fact that Trigonometry is the science of the angle and the angle only, and begins with the general definitions of the *ratios*. He says in the preface that " it is customary to define the trigonometrical ratios by means of a right-angled triangle ; this encourages the mistaken idea that the ratios are fundamentally attached to a triangle." While not denying the correctness of this view we have to point out that, according to the general experience of teachers, the beginner finds solid ground in the right-angled triangle to take his first jump off into theoretical trigonometry.

There are numerous examples of varied types most of which are taken from recent Army Entrance, Cambridge and Oxford School, and Cambridge and Oxford Local Examination Papers.

Geometry for Schools, Vols. I—VI : by W. G. Borchardt and A. D. Perrot, (London, Bell & Sons, 1913.)

Volumes I—III of this treatise have been reviewed in the *Journal* for August 1912. The additional volumes cover the ground of Euclid III, IV, VI, XI. The deductive proofs of the propositions in this portion are given in Theorems 27—93 and in Constructions 14—30. A large amount of useful matter, treating of easy Modern Geometry and the more difficult Problems, is given in the supplementary Theorems $(\text{Y}-\text{I}_a)$ and the Addenda to the several volumes. The numerous Exercises are carefully chosen and the Test Papers now run up to 140.

As before the theoretical treatment is on the whole simple, clear and adequate. The figures are neat and instructive, especially in the portion treating of Plans and Solids which contains at the end valuable hints on the construction of simple Plans and Elevations. The practical treatment of the Circle and the Plane appears to be rather meagre, like that of Areas as pointed out in the former review.

K. J. S.

SHORT NOTES.

A Special Tetrahedron.

The tetrahedron in which the products of opposite edges are equal has some interesting properties. Professor Wilkinson has discussed some of these analytically on pp. 115-7 of the present volume of the J. I. M. S. Geometrical proofs are also simple enough as may be gathered from the following note.

[*Notation* : We shall denote BC, CA, AB, AD, BD, CD by a, b, c, a', b', c' respectively ; so that

$$aa' = bb' = cc' = k^2 \text{ say}.$$

K_1, K_2, K_3, K_4 denote the symmedian points of the faces opposite to A, B, C, D ; and AK_1 is the *symmedian* through A.

A plane parallel to the tangent plane at A to the circumsphere (centre O) cuts the edges through A in points forming an *anti-parallel* triangle. Antiparallels with respect to A, B, C, D will be denoted by $B_1C_1D_1$, $A_2C_2D_2$, $A_3B_3D_3$, $A_4B_4C_4$.]

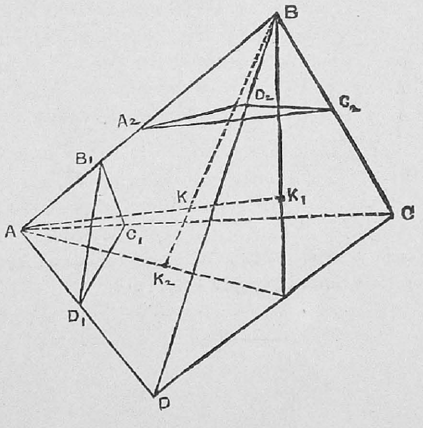
1. *All the four systems of anti-parallels of the tetrahedron are equilateral triangles.*

For, the sides of the antiparallels are proportional to aa', bb', cc' as proved in Casey's *Sphl. Trig.*, § 126. These are equal in the special tetrahedron. Hence the result.

Also, if $DA.DA_4 = \lambda_4^2$, we find that $B_4C_4 = \lambda_4^2.aa'/a'b'c'$.

2. *The lines AK_1. BK_2, CK_3, CK_4, are concurrent.*

We know from plane geometry that AK_2 divides CD in the ratio $AD^2 : AC^2$; and BK_1 divides CD in the ratis $BD^2 : CD^2$. Since these ratios are equal in this tetrahedron, AK_2, BK_1 meet CD in the same point ; and therefore AK_1 and BK_2 intersect.



Let the point of intersection be K, the symmediam point of the tetrahedron, and let its tetrahedral co-ordinates be $\alpha, \beta, \gamma, \delta$. Then because AK meets the face BCD in K_1 which is the symmedian point of BCD, we have

$$\beta : \gamma : \delta = CD^2 : BD^2 : BC^2 = c'^2 : b'^2 : a^2.$$

Also, from BK,

$$\gamma : \delta : \alpha = a'^2 : b^2 : c'^2$$

$$\therefore \alpha : \beta : \gamma : \delta = ab'c' : bc'a' : ca'b' : abc, \text{ since } aa' = bb' = cc'.$$

The symmetry of the above result shows that K lies also on CK_3 and DK_4.

3. *The symmedians AK_1, &c., are the loci of the centroids of the corresponding antiparallels.*

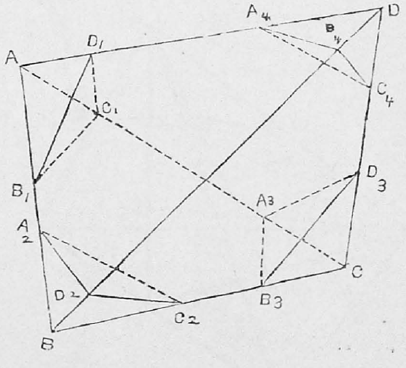
For, the symmedian AK_2 in the triangle ACD bisects the antiparallel C_1D_1. Similarly AK_3, AK_4 bisect B_1D_1 and B_1C_1. Thus AK_1 which is the common section of the planes ABK_2, ACK_3, ADK_4, passes through the centre of medians of $B_1C_1D_1$; in other words, AK_1 is the locus of the centroid of the antiparallel $B_1C_1D_1$.

4. *If four antiparallels of different systems be equal, their angular points lie on a sphere.* (Tucker Sphere, say)

Let $B_1C_1D_1$, $A_2C_2D_2$, $A_3B_3D_3$ $A_4B_4C_4$ be the four antiparallels. If the vertices should lie on a sphere, four sets of conditions should hold, of which one is

$$DD_1.DA_4 = DD_2.DB_4 = DD_3.DC_4.$$

But $\quad DA_4 = \lambda_4^2/a'$; $DD_1 = a' - AD_1 = a' - \lambda_1^2/a'$.



Substituting $\frac{\lambda_4^2}{a'}\left(a' - \frac{\lambda_-^2}{a'}\right) = .\&c., = \&c.$

$\therefore$ $$\frac{\lambda_1}{a'} = \frac{\lambda_2}{b'} = \frac{\lambda_3}{c'}$$

$$\frac{\lambda_4}{a'} = \frac{\lambda_2}{c} = \frac{\lambda_3}{b}$$

Similarly $$\frac{\lambda_1}{c} = \frac{\lambda_3}{a} = \frac{\lambda_4}{b'}$$

$$\frac{\lambda_4}{c'} = \frac{\lambda_1}{b} = \frac{\lambda_2}{a}.$$

The above conditions are obviously satisfied, when $aa' = bb' = cc'$; and then

$$\lambda_1^2 : \lambda_2^2 : \lambda_3^2 : \lambda_4^2 = bca' : cab' : abc' : abc.$$

The sides of the antiparallels are

$$\frac{k^2\lambda_1^2}{bca'}, \frac{k^2\lambda_2^2}{cab'}, \frac{k^2\lambda^{32}}{abc'}, \frac{k^2\lambda_4^2}{abc}.$$

and are therefore equal. Hence the proposition.

5. *The centre T of the above sphere lies on KO.*

For, by § 3, AK passes through the centre of the equilateral triangle $B_1C_1D_1$, say O_1; and OA is perpendicular to the antiparallel $B_1C_1D_1$. Hence TO_1 is parallel to OA and in the same plane with it. That is, the line through C_1 parallel to OA meets OK in T.

6. *The vertices of the antiparallel triangles passing through K all lie on a sphere whose centre is K.* (Analogue of the Cosine Circle)

This is a particular case of the above.

7. *The parallels through K to the faces of the tetrahedron meet the edges in twelve points lying on a sphere.* (Analogue of the Lemoine Circle)

Let the parallels through K $(\alpha\beta\gamma\delta)$ be $A_2A_3A_4$, $B_1B_3B_4$, $C_1C_2C_4$, $D_1D_2D_3$. In order that these twelve points may lie on a sphere, four sets of conditions should hold of which one is

$$DD_1.DA_4 = DD_2.DB_4 = DD_3.DC_4.$$

But $DD_1 = a'(1-\delta)$, and $DA_4 = a'\alpha$; etc.

$\therefore$ $$a'^2\alpha(1-\delta) = b'^2\beta(1-\delta) = c'^2\gamma(1-\delta)$$

i.e., $$\alpha : \beta : \gamma = 1/a'^2 : 1/b'^2 : 1/c'^2$$
$$= a^2 : b^2 : c^2,$$

which is the case.

Hence the theorem.

8. *The centre L of the above sphere lies on OK.*

For, the above sphere may be regarded as a Tucker sphere corresponding to the antiparallels $B_1C_1D_1$, &c., obviously.

. Therefore, by § 5, L lies on OK.

9. *The locus of the vertex D of the special tetrahedron on a given base ABC is a circle whose plane is perpendicular to the base.*

This follows immediately since DB : DC $= c : b$, and therefore D lies on the Apollonian sphere corresponding to B, C. Similarly, D lies on the Apollonian sphere corresponding to A, C. That is, the locus required is the common section of these two spheres which is perpendicular to ABC.

10. *The line OK meets the four altitudes of the tetrahedron.*

For, the circle in § 9 meets the base ABC in the *isodynamic points* which are collinear with the circumcentre and the symmedian point of ABC (Gallatly's *Geometry*, § 26).

Also, the plane of the circle is perpendicular to ABC, and therefore the foot of the perpendicular from D on ABC is also on the line of collinearity.

Thus, the plane of the circle contains the altitude and passes through the circumcentre of ABC and K_4. But the point O is on the line through the circumcentre of ABC parallel to the altitude; and K is on DK_4.

Hence OK is in the same plane with the altitude from D. Similarly for the other altitudes.

11. *The lines joining the vertices to the poles of the opposite faces with respect to the circumsphere are concurrent at K.*

This may be seen analytically as follows:

The equation of the circumsphere may be written

$$a^2\beta\gamma + b^2\gamma\alpha + c^2\alpha\beta + a'^2\alpha\delta + b'^2\beta\delta + c'^2\gamma\delta = 0.$$

The pole of ABC, or $\delta = 0$, with respect to the sphere is found from

$$b'^2\beta - a'^2\alpha = 0$$

$$c'^2\gamma - b'^2\beta = 0$$

$$a'^2\alpha - c'^2\gamma = 0$$

i.e. $$\alpha a'^2 = \beta b'^2 = \gamma c'^2$$

or $$\alpha : \beta : \gamma = a^2 : b^2 : c^2,$$

showing that the join of D to the pole passes through K.

N. DURAIRAJAN.

On Factors.

1. Let $x=a+ib$, then, if X be any function of x, we may write $X=A+iB$; so that, rationalizing the equations, we infer that if $(x-a)^2+b^2=0$, then $(X-A)^2+B^2=0$.

In other words, $[x-a)^2+b^2]$ is a factor of $[(X-A)^2+B^2]$.

Trigonometrically the result may be expressed in the following form:

If $x=r(\cos\alpha+i\sin\alpha)$, then $X=R(\cos\theta+i\sin\theta)$ where θ is a known function of α; so that $(x^2-2xr\cos\alpha+r^2)$ is a factor of $(X^2-2XR\cos\theta+R^2)$.

2. If $X\equiv x^n$, the expression $[(X-A)^2+B^2]$ is of the $2n^{th}$ degree, and its remaining $(n-1)$ quadratic factors are obtained by replacing x by $x\omega$, $x\omega^2$, $x\omega^3$,......$x\omega^{n-1}$ in $[(x-a)^2+b^2]$, where ω is a primitive n^{th} root of unity. Remembering that imaginary factors occur in pairs, these $(n-1)$ quadratic factors, which are imaginary, may be replaced by $(n-1)$ others, which are real.

Thus, if $\omega=\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}$, the factors corresponding to ω^p, ω^{n-p} are $(x\omega^p-a\pm bi)(x\omega^{n-p}-a\mp bi)$; and these can be re-written

$$(x\cos\frac{2p\pi}{n}-a)^2+(x\sin\frac{2p\pi}{n}\pm b)^2.$$

i.e.,

$$x^2+a^2+b^2-2x\left((a\cos\frac{2p\pi}{n}\mp b\sin\frac{2p\pi}{n}\right).$$

3. If the value of $\cos\alpha$ in § 1 is a quadratic surd say $c+\sqrt{d}$, the, corresponding *rational* factor of the fourth degree is of the form

$$[(x^2-2xra+r^2)^2-4x^2r^2d.]$$

Example: x^8-4x^4+16. (*Vide*: Q 375 on p. 118, J.I.M.S., Vol. IV).

This is of the form $x^{2n}-2x^nr^n\cos n\alpha+r^{2n}$, where $n=4$, $r=\sqrt{2}$, $\cos 4\alpha=\frac{1}{2}$. Hence $\cos\alpha=\pm(\sqrt{3}\pm1)/2\sqrt{2}$.

Thus the factors are $x^2\pm x(\sqrt{3}\pm1)+2$, which lead to the rationalised forms

$$[(x^2\pm x+2)^2-3x^2.]$$

4. Suppose $x=a+b\omega+c\omega^2$, where $\omega^3=1$; then $X=A+B\omega+C\omega^2$ on reduction.

Rationalizing, we infer that

$$(a-x)^2+b^2+c^2-b(a-x)-c(a-x)-bc$$

is a factor of

$$(A-X)^2+B^2+C^2-B(A-X)-C(A-X)-BC.$$

Or $$(a-x)^3+b^3+c^3-3(a-x)bc$$

is a factor of

$$(A-X)^3+B^2+C^3-3(A-X)BC.$$

The remaining factors are written as in §2.

5. Generally, if $x=a_1+a_2\omega+a_3\omega^2+\dots\dots\dots a_n\omega^{n-1}$,

then $$X=A_1+A_2\omega+A_3\omega^2+\dots\dots\dots A^n\omega^{n-1},$$

where ω is a primitive n^{th} root of unity.

From the rationalized forms of these equations we infer that the determinant

$$\begin{vmatrix} a_1-x, & a_2, & a_3, & \dots & \dots & \dots & a_n \\ a_n\ , & a_1-x, & a_2, & \dots & \dots & \dots & a_{n-1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_2 & a_3, & a_4, & \dots & \dots & \dots & a_1-x \end{vmatrix}$$

is a factor of the corresponding determinant obtained by replacing the small letters x, a_1, a_2,.........a_n by the capitals X_1 A_1, A_2,.........A_n.

M. T. NARANIENGAR.

The Face of the Sky for November and December 1913.

	November.				December.		
	H.	M.	S.		H.	M.	S.
1	22	41	4	...	0	39	21
8	23	8	40	...	1	6	57
15	23	36	16	...	1	34	33
22	0	3	52	...	2	2	9
29	0	31	28	...	2	29	45

From this table the constellations visible during the evenings of November and December can be ascertained by a reference to the positions as given in a star atlas.

Phases of the Moon.

		November.					December.			
		D.	H.	M.			D.	H.	M.	
First Quarter	...	6	0	4	A.M.	...	5	8	29	P.M.
Full Moon	...	14	4	41	,,	...	13	8	30	,,
Last Quarter	...	21	1	26	P.M.	...	20	9	46	,,
New Moon	...	28	7	11	A.M.	...	27	8	29	,,

The Planets.

Mercury which was in superior conjunction on September 16, attains its greatest elongation (23° 34′ E) on November 2nd and is stationary on November 13th and reaches inferior conjunction on November 23rd. It is again stationary on December 2nd and attains its greatest elongation (21° 1′ W) on December 11th. It is in conjunction with the moon on November 27th, and December 26th and with Venus on December 3rd.

Venus is a morning star. It is in conjunction with the moon on November 30th and on December 26th.

Mars is stationary on November 27th. It is in conjunction with the moon on November 19th at 12-36 a. m. and on December 16th at 3-27 a. m.

Jupitor (which was stationary on September 4th), is in conjunction with the moon on December 28th.

Saturn is in opposition on December 27th. It is in conjunction with the moon on November 16th and December 13th.

Uranus is in conjunction with the moon on November 5 and December 2 and 30. Neptune is in conjunction with the moon on December 16th.

V. Ramesam.

SOLUTIONS.

Question 380.

(A. C. L. WILKINSON, M. A.) :—At the extremity A of the major (or minor) axis of a conic S, the circle of curvature C is drawn. PQ is any chord of the conic S which touches C, and T is its pole with respect to the conic S. A conic S′ is described through PQT touching the conic S at A. Show that the tangents at P,Q to S′ also touch the circle C, and that the radius of curvature of the conic S′ at A is half the radius of the circle C.

Solution by the Proposer.

Let T' be the pole of PQ with respect to S'. Project P, Q into the circular points at infinity. Denote by small letters the corresponding points of the projection. S becomes a circle s centre t, C becomes a parabola c having 4-point contact with s, *viz.*, s is the circle of curvature at the vertex a of the parabola, hence s is of radius $2a$ and centre t, S' becomes a circle s' passing through t and touching the parabola at the vertex a, hence t' the centre of s' is at the focus of the parabola. The straight lines joining t' to the circular points at infinity are tangents both to c and s', therefore the tangents at P, Q to S' also touch the circle C. If two conics given by $\sqrt{\lambda\alpha}+\sqrt{\mu\beta}+\sqrt{\nu\gamma}=0$, and $\alpha^2-k\beta\gamma=0$ touch each other, their radii of curvature at the point of contact are in the ratio 2 : 1 by the general formula given on p. 182, J. I. M. S. Vol. 5; thus taking the triangle formed by the tangents at P and Q to S' and their chord of contact as the triangle of reference the second result follows at once.

Question 458.

(K. J. SANJANA M.A.) :—Sum the series—

$$S=\frac{\cos\theta}{1}+\frac{1}{2}\cdot\frac{\cos^3\theta}{3}+\frac{1.3}{2.4}\,\frac{\cos^5\theta}{5}+\frac{1.3.5}{2.4.6}\,\frac{\cos^7\theta}{7}+\ldots\ldots\ldots,$$

$$T=\frac{\cos^2\theta}{2}+\frac{2}{3}\cdot\frac{\cos^4\theta}{4}+\frac{2\cdot4}{3\cdot5}\,\frac{\cos^6\theta}{6}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\,\frac{\cos^8\theta}{8}+\ldots\ldots\ldots,$$

and prove that

$$\tfrac{1}{2}\pi S-T=\frac{4}{\pi}\left\{\frac{\cos\theta}{1^3}-\frac{\cos 3\theta}{3^3}+\frac{\cos 5\theta}{5^3}-\ldots\ldots\ldots\right\}.$$

Solution by G. Ramachandran, T. P. Trivedi, R. Srinivasan, and H. V. Venkataramiengar.

We have the well-known series

$$\sin^{-1}x=x+\tfrac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot3}{2\cdot4}\,\frac{x^5}{5}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\,\frac{x^7}{7}+\ldots\ldots\ldots$$

and

$$\tfrac{1}{2}(\sin^{-1}x)^2=\frac{x^2}{2}+\frac{2}{3}\cdot\frac{x^4}{4}+\frac{2\cdot4}{3\cdot5}\cdot\frac{x^6}{6}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\cdot\frac{x^8}{8}+\ldots\ldots$$

Put $x=\cos\Theta$ in these ; then

$$S=\sin^{-1}(\cos\Theta)=\frac{\pi}{2}-\Theta \quad \ldots \quad \ldots \quad \ldots \quad (1),$$

$$T=\tfrac{1}{2}\{\sin^{-1}(\cos\Theta)\}^2=\tfrac{1}{2}\left(\frac{\pi}{2}-\Theta\right)^2 \quad \ldots \quad \ldots \quad (2),$$

Again

$$\frac{\cos\Theta}{1^3}-\frac{\cos 3\Theta}{3^3}+\frac{\cos 5\Theta}{5^3}\ldots\ldots\ldots=\frac{\pi}{8}\left(\frac{\pi^2}{4}-\Theta^2\right);$$

[Ex. 12 page 86, Edwards' *Diff. Cal.*]

$$\text{Hence, } \tfrac{1}{2}\pi S-T=\tfrac{1}{2}\pi\left(\frac{\pi}{2}-\Theta\right)-\tfrac{1}{2}\left(\frac{\pi}{2}-\Theta\right)^2$$

$$=\tfrac{1}{2}\left(\frac{\pi^2}{4}-\Theta^2\right)=\frac{4}{\pi}\times\frac{\pi}{8}\left(\frac{\pi^2}{4}-\Theta^2\right)$$

$$=\frac{4}{\pi}\left\{\frac{\cos\Theta}{1^3}-\frac{\cos 3\Theta}{3^3}+\frac{\cos 5\Theta}{5^3}-\ldots\ldots\ldots\right\}$$

Question 459.

(S. KRISHNASWAMI AIYANGAR) :—If $a_r=\sum_{1}^{8}\left(\frac{n^r}{\lfloor n}\right)$ prove that

$$a_{r+1}=a_r+ra_{r-1}+\frac{r.r-1}{\lfloor 2}a_{r-2}+\ldots ra_1+a_0.$$

Solution by G. Ramachandran.

We have $a_r+ra_{r-1}+\frac{r.r-1}{\lfloor 2}a_{r-2}+\ldots\ldots\ldots+ra_1+a_0$

$$=\sum_{1}^{n=\infty}\left(\frac{n^r}{\lfloor n}\right)+r\sum_{1}^{n=\infty}\left(\frac{n^{r-1}}{\lfloor n-1}\right)+\frac{r.r-1}{\lfloor 2}\sum_{1}^{n=\infty}\left(\frac{n^{r-2}}{\lfloor x-2}\right)+\ldots\ldots$$

Now the terms on the right hand side are respectively the coefficients of x^r in

$\lfloor r\ e^{e^x}$, $\lfloor r-1\ x.e^{e^x}$; $\lfloor r-2 .x^2\ e^{e^x}$, etc., etc.

$\therefore \quad a_r+ra_{r-1}+\ldots\ldots+ra_1+a_0$

$$=\text{coefficient of } x^r \text{ in } \lfloor r\ e^{e^x}\left\{1+x+\frac{x^2}{\lfloor 2}+\frac{x^3}{\lfloor 3}+\ldots\right\}$$

$$=\text{coefficient of } x^r \text{ in } \lfloor r\ e^{e^x}.e^x$$

$$=\text{coefficient of } x^r. \text{ in } \underline{|r} \left\{ e^x+e^{2x}+\frac{e^{3x}}{\underline{|2}}+\frac{e^{4x}}{\underline{|3}}+\dots \right\}$$

$$=1^r+2^r+\frac{3^r}{\underline{|2}}+\frac{4^r}{\underline{|3}}+\dots\dots$$

$$=1^{r+1}+\frac{2^{r+1}}{\underline{|2}}+\frac{3^{r+1}}{\underline{|3}}+\frac{2^{r+1}}{\underline{|3}}+\dots\dots$$

$$=\sum_{1}^{n=\infty}\frac{n^{r+1}}{\underline{|r+1}}$$

$$=\quad a_{r+1}.$$

Question 460.

(V. V. Satyanarayan):—Find a point such that the feet of the perpendiculars from it on the sides of a given quadrilateral may form a parallelogram. Discuss the number of solutions.

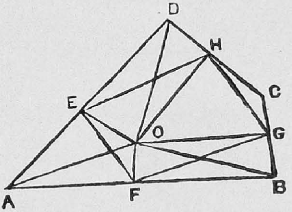
Solution (1) by N. Sankara Aiyar, (2) by the Proposer.

(1) Let O be the required point.

Then FE = OA sin A, GH = OC sin C,

FG = OB sin B, EH = OD sin D.

If FE = GH and FG = EH, then EFGH will be a parallelogram.



$\therefore$ OA sin A = OC sin C

and OB sin B = OD sin D.

Hence to find O, describe the circles which are the loci of points satisfying the above conditions. They will intersect in two points, real or imaginary.

In the cyclic quadrilateral the circumcentre is the only point.

(2) Join OB, OD. Then, in the cyclic quadrilaterals OHDE, OFBG, $\underline{|DEH} = \underline{|DOH}$ and $\underline{|BFG} = \underline{|BOG}$.

Also $\underline{|GOH} = 180^\circ - C$,

$\therefore$ $\underline{|BOD} = 180 - C + \underline{|DEH} + \underline{|BFG}$.

But $\lfloor DEH + \lfloor BFG = A$, since EH and FG are parallel.

Hence $\lfloor BOD = 180^\circ + A - C.$

Similarly $\lfloor AOC = 180^\circ + D - B.$

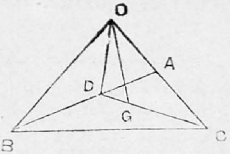
The point O is thus determined as the intersection of known circles on BD and AC, and there are consequently two solutions.

Question 461.

(Selected) :—Prove that the dihedral angle of a regular tetrahedron is the supplement of that of a regular octahedron.

Solution by H. V. Venkataramiengar, K. J. Sanjana, M.A., and T. P. Trivedi, M.A., L.L.B.

Let OABC be the regular tetrahedron ; G the centroid of the triangle ABC. Thus OG is perpendicular to the plane ABC.



Let D be the middle point of AB ; OD is perpendicular to AB in the plane OAB and CD is perpendicular to AB in the plane ABC. Hence $\angle ODC$ ($\equiv \theta$) is the dihedral angle of the tetrahedron ABC. Now G being the centroid of the triangle ADC, $DG = \frac{1}{3}DC$; but $DC = OD$, being medians of equal equilateral triangles ; hence $DG = \frac{1}{3}OD$ and $\cos \theta = \frac{1}{2}$.

An octahedron may be looked upon as made up of two equal pyramids constructed upon the square base EFGH and on opposite sides of it.

Let O′EFGH be half the octahedron ; K the centroid of EFGH and L the mid point of EH. Now O′L is perp to EH in the plane O′EH and K L is perpendicular to EH in the plane EFGH. Hence O′LK ($\equiv \phi$) is half the dihedral angle of the regular octahedron, $KL = \frac{1}{2}a$ and $O'L = \frac{\sqrt{3}}{2}a$; also O′K is perp. to KL.

Hence $$\cos \phi = \frac{KL}{O'L} = \frac{1}{\sqrt{3}};$$

$$\therefore \quad \cos 2\phi = 2\cos^2\phi - 1 = -\tfrac{1}{3} = -\cos\theta;$$

$$\therefore \quad 2\phi = 180 - \theta.$$

Question 464.

(S. RAMANUJAN) :—$(2^n—7)$ is a perfect square for the values 3, 4, 5 7, 15 of n. Find other values.

Solution by K. J. Sanjana and T. P. Trivedi.

Put $2^n-7=x^2$.

(1) Let n be even and $=2m$; then $2^{2m}-x^2=7$.

$\therefore \qquad (2^m—x)(2^m+x)=7.$

The only integral factors of 7 are 1 and 7;

$\therefore \quad 2^m-x=1$ and $2^m+\pm x=7$.

Hence $x=3$, and $2^m=4$ or $m=2$; and $n=4$, which is the only even integral value of n.

(2) Let n be odd and $=2m+1$; then $2^{2m+1}—x^2=7$,

$\therefore \quad x^2—2\cdot 2^{2m}=—7$; if $y=2^m$, we get $x^2—2y^2+7=0$. ... (1)

We have to solve this Pellian equation and pick out those values of y which are powers of 2.

Reduce $\sqrt{2}$ to a continued fraction and find the convergents:

$$\frac{1}{1}, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \frac{239}{169} \text{ etc.}$$

These give $x^2—2y^2\pm1=0$, alternately; hence we take $\frac{1}{1}, \frac{7}{5}, \frac{41}{29}, \frac{239}{169}$ etc., for $\frac{x}{y}$.

By Lagrange's method (Chrystal's *Alg.* § 18), 7 may be reduced to a number not greater than 3 ($\frac{7}{2}=3\frac{1}{2}$). We have $\frac{K_1^2-C}{H}$ an integer K_1 being not greater than $\frac{1}{2}$ H; here $H=—7, C=2$, $\therefore \frac{K_1^2-2}{—7}=$ integer.

This gives $K_1=3$ which makes $\frac{K_1^2-2}{-7}=-1=H_1$.

Now H_1 is $<\sqrt{2}$; $\therefore$ we solve $x_1^2-2y_1^2=-1$;

Then $x=\frac{3x_1\pm 2y_1}{-1}$, $y=\frac{3y_1\mp x_1}{-1}$ is a solution of (1).

As x and y are squared, $x=3x_1\mp 2y_1$ and $y=3y_1\mp x_1$, and the numerical values of the latter should be powers of 2.

Thus

$y=3\cdot1\mp1=2$ or 4 $\therefore m=1$ or 2 and $n=3, 5$;

$y=3\cdot5\mp7=8$ or 22 $\therefore m=3$ and $n=7$;

$y=3\cdot29\mp41=46$ or 128 $\therefore m=7$ and $n=15$;

$y=3\cdot169\mp239=268$ or 746; no solution;

$y=3\cdot985\mp1393=1562, 4348$; no solution; and so on.

It is found on actual calculation that no value of n up to 75 satisfies the proposed equation.

A general formula for y may be obtained from the recurrence values of x_1, y_1; but as the scales (both $1-6x+x^2$) are not factorisable, such a formula is not convenient for numerical calculation.

Question 467.

(N. Sankara Aiyar M. A) :—If O is the circumcentre of ABC and A′B′C′ the triangle whose sides bisect OA, OB, OC perpendicularly, prove that AA′, BB′, CC′ are concurrent.

Solution by N. P. Pandya and V. V. Satyanarayan.

Since A′ is the circumcentre of $\triangle$OBC,

$$BA'=A'O=A'C$$

and $$\angle A'BO=\angle BOA'=\tfrac{1}{2}\angle BOC=\angle A,$$

also $$\angle OBA=90^\circ-C.$$

$\therefore$ $$\angle A'BA=90^\circ-C+A.$$

Similarly $$\angle ACA'=90^\circ-B+A.$$

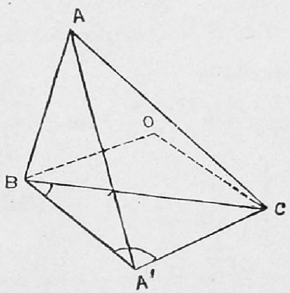
Now $$\frac{\sin BAA'}{\sin A'BA}=\frac{BA'}{A'A}=\frac{A'C}{A'A}=\frac{\sin A'AC}{\sin A'CA}$$

$\therefore$ $$\frac{\sin BAA'}{\sin A'AC}=\frac{\sin A'BA}{\sin A'CA}=\frac{\cos(C-A)}{\cos(A-B)}$$

Similarly $$\frac{\sin CBB'}{\sin B'BA}=\frac{\cos(A-B)}{\cos(B-C)}$$

and $$\frac{\sin ACC'}{\sin C'CB}=\frac{\cos(B-C)}{\cos(C-A)}$$

$\therefore$ $$\frac{\sin BAA' \sin CBB' \sin ACC'}{\sin A'AC \sin B'BA \sin C'CB}=1.$$



Hence AA′, BB′, CC′ are concurrent.

Question 470.

(S. KRISHNASWAMI AIYANGAR) :—If ρ, ρ' be the radii of curvature at corresponding points of a curve and its evolute prove that

$$\rho'/\rho = \{ 3y_1 y_2^2 - y_3(1+y_1^2) \} / y_2^2.$$

Solution by R. Srinivasan, M.A. and others.

We know $$\rho = \frac{(1+y_1^2)^{\frac{3}{2}}}{y_2}.$$

$$\rho' = \frac{d\rho}{d\psi} = \frac{ds}{dx} \cdot \frac{dx}{ds} \cdot \frac{ds}{d\psi}.$$

$$\therefore \quad \frac{\rho'}{\rho} = \frac{1}{\sqrt{1+y_1^2}} \cdot \frac{ds}{dx}$$

$$= \frac{1}{\sqrt{1+y_1^2}} \cdot \frac{\frac{3}{2} \cdot (1+y_1^2)^{\frac{1}{2}} \cdot 2y_1 y_2^2 - (1+y_1^2)^{\frac{3}{2}} y_3}{y_2^2}$$

$$= \frac{3y_1 y_2^2 - y_3(1+y_1^2)}{y_2^2}$$

Question 471.

(K. J. SANJANA) :—Prove that

$$\frac{1}{2}\frac{x^2}{2} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^4}{4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^6}{6} \ldots\ldots = \log \frac{2+2\sqrt{1-x^2}}{x^2},$$

and find the sum of

$$\frac{2}{3}\frac{x^3}{3} + \frac{2 \cdot 4}{3 \cdot 5}\frac{x^5}{5} + \frac{2 \cdot 4 \cdot 6}{3 \cdot 5 \cdot 7}\frac{x^7}{7} \ldots\ldots\ldots\ldots$$

Solution of by S. Ramanujan, and N. Sankara Aiyar.

The question should be

$$\frac{1}{2} \cdot \frac{x^2}{2} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^4}{4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^6}{6} + \&c = \log \frac{2}{1+\sqrt{1-x^2}}.$$

The left side $= \displaystyle\int \left(\frac{1}{\sqrt{1-x^2}} - 1 \right) \frac{dx}{x}$ (Substitute $\sin 2\theta$ for x).

$$= 2\int \tan \theta \, d\theta = -2 \log \cos\theta.$$

$$= \log \frac{2}{1+\cos 2\theta} = \log \frac{2}{1+\sqrt{1-x^2}}.$$

Also,

$$x+\frac{2}{3}\cdot\frac{x^3}{3}+\frac{2\cdot4}{3\cdot5}\cdot\frac{x^5}{5}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7}\cdot\frac{x^7}{7}+\&c.$$

$$=\int_0^x \frac{\sin^{-1}y}{y\sqrt{1-y^2}}dy \quad \text{(Substitute } \sin\theta \text{ for } y\text{)}$$

$$=\int_0^{\sin^{-1}x} \frac{\theta}{\sin\theta}d\theta. \quad \text{(Put } \tan\frac{1}{2} \text{ equal to } z$$

$$=2\int_0^{\frac{1}{2}\theta}\frac{1}{z}\cdot\tan^{-1}z\,dz$$

Question 472.

(Selected) :—Evaluate $\int_0^\infty \frac{\cos x\,dx}{(1+x^2)^2}, \int_0^\infty \frac{\cos 2x\,dx}{(1+x^2)^2}$.

Solution (1) *by T. P. Trivedi, M.A., LL.B., and K. J. Sanjana, M.A.* ; (2) *by R. Srinivasan, M.A., and S. Ramanujan* ; (3) *by J. M. Bose, M.A., B.Sc.*

(1) Consider the more general form $\int_0^\infty \frac{\cos rx\,dx}{(1+x^2)^2}\equiv B$, say.

Differentiating twice for r we have

$$\frac{d^2B}{dr^2}=-\int_0^\infty \frac{x^2\cos rx\,dx}{(1+x^2)^2}=-\int_0^\infty\frac{\cos rx\,dx}{1+x^2}+\int_0^\infty\frac{\cos rx\,dx}{(1+x^2)^2};$$

$$\therefore \qquad \frac{d^2B}{dr^2}-B=-\int_0^\infty\frac{\cos rx\,dx}{1+x^2}=-\frac{\pi}{2}e^{-r}, \text{ as is well known.}$$

Solving this differential equation, we have

$$B=c_1e^2+c_2e^{-r}-\frac{\pi}{4}\left\{\frac{e^{-r}}{D-1}-\frac{e^{-r}}{D+1}\right\}$$

$$=c_1e^r+c_2e^{-r}+\frac{\pi}{8}e^{-r}+\frac{\pi}{4}re^{-r},.$$

We have now to determine the constants c_1,c_2. As $\cos rx$ is finite when $r=\infty$, the integral also remains finite then ; thus c_1 must vanish.

When $r=o$, the integral$=\int_0^\infty\frac{dx}{(1+x^2)^2}=\int_0^{\frac{1}{2}\pi}\cos^2\theta d\theta$, if $x=\tan\theta$;

thus $$c_2+\frac{\pi}{8}=\frac{1}{2}\cdot\frac{\pi}{2}, \text{ so that } c_2=\frac{\pi}{8}.$$

Hence finally $B=\frac{\pi}{4}e^{-r}(1+r)$; so that the two given integrals are equal to $\frac{\pi}{2e}$ and $\frac{3\pi}{4e^2}$ respectively.

Similarly, putting $C=\int_0^\infty \frac{\cos rx\, dx}{(1+x^2)^3}$, we shall have

$$\frac{d^2C}{dr^2}-C=-B=-\frac{\pi}{4}e^{-r}(1+r);$$

whence by the same method as above we shall get $\frac{\pi}{16}e^{-r}(3+3r+r^2)$ as the value of C. And D, E, F...may be successively derived in the same manner.

(2) We know that $\int_0^\infty \frac{\cos rx}{1+x^2}dx=\frac{\pi}{2}e^{-r}$. [Williamson, p. 151, (39).]

Changing x into $\frac{y}{c}$, we get

$$\int_0^\infty \frac{\cos ry/c}{c^2+y^2}dy=\frac{\pi}{2}.\frac{e^{-r}}{c}.$$

or, putting a for $\frac{r}{c}$ we get

$$\int_0^\infty \frac{\cos ay}{c^2+y^2}dy=\frac{\pi}{2}.\frac{e^{-ac}}{c}.$$

Differentiating both sides with regard to c, we get

$$\int_0^\infty \frac{\cos ay}{(c^2+y^2)^2}dy=\frac{\pi}{4}e^{-ac}.\frac{1+ac}{c^3}$$

Putting $c=a=1$, we get

$$\int_0^\infty \frac{\cos y}{(1+y^2)^2}dy=\frac{\pi}{2}e^{-1}.$$

Putting $c=1$, $a=2$ we get

$$\int_0^\infty \frac{\cos 2y}{(1+y^2)^2}dy=3\frac{\pi}{4}e^{-2}$$

(3) The value of these integrals can be obtained from the more general integral $\int_0^\infty \frac{\cos px}{(1+x^2)^2}dx$.

Now $\int_0^\infty \frac{\cos px}{(1+x^2)^2}dx=\frac{1}{2}\int_{-\infty}^{+\infty}\frac{\cos px}{(1+x^2)^2}$, since the integrand is an even function.

The required integral is the real part of $\frac{1}{2}\int\frac{e^{pzi}}{(1+z^2)^2}dz$ taken round a suitable contour C.

Take a contour on the upper half of z-plane consisting of a semi-circle (Γ) of large radius and the real axis from $-\infty$ to $+\infty$.

$\int \frac{e^{pzi}}{(1+z^2)^2} dz = 2\pi i \times$ the sum of the residues of the poles of the function within the contour. Now at all points on a large semicircle, the integrand is equal to $\frac{e^{pzi}}{z^4}$ which vanishes in the limit since e^{pzi} remains finite.

$$\text{Hence} \int_c = \int_\Gamma + \int_{-\infty}^{+\infty} = \int_{-\infty}^{+\infty}, \text{ since} \int_\Gamma \text{ is zero.}$$

Now the function has a single pole $z=i$ of the second order within the contour, and the residue at $z=i$ is given by

$$R = \underset{z=i}{L_t} \frac{d}{dz} \left(\frac{e^{ipz}}{(z+i)^2} \right)$$

Differentiating and putting $z=i$, we get

$$R = -\tfrac{1}{4}\, ie^{-p}(p+1)$$

$$\therefore \qquad \int_{-\infty}^{+\infty} \frac{e^{pzi}}{(1+z^2)^2} dz = 2\pi i \times -\tfrac{1}{4}\, ie^{-p}(p+1)$$

$$\frac{\pi}{2} = e^{-p}(p+1).$$

Equating the real and imaginary parts

$$\int_{-\infty}^{+\infty} \frac{\cos px}{(1+x^2)^2} dx = \frac{\pi}{2}\, e^{-p}(p+1)$$

and

$$\int_{-\infty}^{+\infty} \frac{\sin px}{(1+x^2)^2} dx = 0.$$

$$\therefore \qquad \int_0^\infty \frac{\cos px\, dx}{(1+x^2)^2} = \tfrac{1}{2} \int_{-\infty}^{+\infty} \frac{\cos px}{(1+x^2)^2} dx = \frac{\pi}{4} e^{-p}(p+1).$$

Putting $p=1$, $$\int_0^\infty \frac{\cos x}{(1+x^2)^2} dx = \frac{\pi}{2e}$$

Putting $p=2$, $$\int_0^\infty \frac{\cos 2x}{(1+x^2)^2} dx = \frac{3\pi}{4e^2}.$$

Question 473.

(A. C. L. Wilkinson M. A., F.R.A.S.) :—PQRS is a spherica. quadrilateral such that PR, and QS are quadrants. If A,B,C, are the intersections of (PR, QS). (PS, QR) and (PQ, RS) respectively, prove that

$$\text{(i)} \cos PA \cos PB = \cos QA \cos QB,$$
$$\text{(ii)} \cos PA \cos PC + \cos QA \cos QC = \cos AC.$$

Solution by K. J. Sanjana, M.A., R. Srinivasan M.A., and H. V. Venkataramiengar.

In a quadrantal triangle XYZ of which $XY = 90°$, we know that $\cos y = -\tan X. \cot Z$ and $\tan y = \tan Y. \operatorname{cosec} X$.

(i) In the $\triangle$ PBR, $\cos PB = -\tan BPR. \cot PBR$;
and in the $\triangle$ QBS, $\cos QB = -\tan BQS. \cot QBS$.

$$\therefore \quad \frac{\cos PB}{\cos QB} = \frac{\tan B\hat{P}R}{\tan B\hat{Q}S} = \frac{\tan S\hat{P}R}{\tan S\hat{Q}R}$$
$$= \frac{\tan S\hat{P}R}{\tan SR} \bigg/ \frac{\tan S\hat{Q}R}{\tan RS} = \frac{\sin PRS}{\sin Q\hat{S}R}$$
$$= \frac{\sin AS}{\sin AR} = \frac{\cos QA}{\cos PA}.$$
$$\therefore \quad \cos PA. \cos PB = \cos QA. \cos QB$$

(ii) In the $\triangle$PCR, we know that

$$\cos PC. \sin AR + \cos RC. \sin AP = \cos AC. \sin PR.$$
$$\therefore \quad \cos PC. \cos PA + \cos RC. \cos AR = \cos AC,$$
i.e. $$\cos PC. \cos PA + \cos QA. \cos QC = \cos AC, \text{ by (i)}.$$

Question 479.

(S. P. Singaravelu Modeliar) :—Shew that the equation

$$(x^5 - 10x^3y^2 + 5xy^4) \cos 5\alpha + (y^5 - 10y^3x^2 + 5yx^4) \sin 5\alpha$$
$$-5ar^4 + 20a^3r^2 - 16a^5 = 0,$$

represents the sides of a regular pentagon ; and find a similar equation for the sides of a regular heptagon.

Solution by K. J. Sanjana, M.A., and T. P. Trivedi, M.A., L.L.B.

More generally, let a regular polygon of n sides, of in radius a, be referred to its centre as origin ; and let the perp. OA_1 on one side make $\angle A_1Ox = \alpha$, with the initial line.

Then $\angle A_2Ox = \alpha + 2\pi/n$, $\angle A_3Ox = \alpha + 4\pi/n, \ldots$

$\angle A_nOx = \alpha + 2(n-1)\pi/ \ldots$

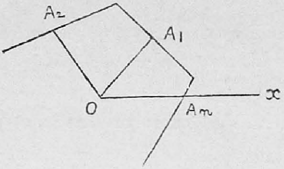
The equations to the various sides are

$x \cos \alpha + y \sin \alpha - a = 0$, $x \cos (\alpha + 2\pi/n) + y \sin (\alpha + 2\pi/n) - a = 0, \ldots$

Reducing to polars, we get

$$r\cos(\theta-\alpha)-a=0,\ r\cos(\theta-\alpha-2\pi/n)-a=0,\ldots$$

$$r\cos\left(\theta-\alpha-\pi\frac{n-1}{n}2\pi\right)-a=0.$$



If any one of the angles is denoted by ϕ, we see that $\frac{a}{r}$ is a root of the equation $\cos n\phi=\cos(n\theta-n\alpha)$; thus

$$2\cos(n\theta-n\alpha)=2\cos n\phi=\left(\frac{2a}{r}\right)^{n}-n\left(\frac{2a}{r}\right)^{n-2}+\frac{n(n-3)}{1\cdot 2}\left(\frac{2a}{r}\right)^{n-4}-\ldots$$

$$+(-1)^r\frac{n(n-r-1)(n-r-2)\ldots(n-2r+1)}{\lfloor r}\left(\frac{2a}{r}\right)^{n-2r}+\ldots$$

Hence $r^n\cos(n\theta-n\alpha)=2^{n-1}a^n-\frac{n}{1}2^{n-3}a^{n-2}r^2+\frac{n(n-3)}{1\cdot 2}2^{n-5}a^{n-4}r^4-\ldots$

and the left side $=r^n\cos n\theta.\cos n\alpha+r^n\sin n\theta.\sin n\alpha$; so that by expanding $\cos n\theta$, $\sin n\theta$ in powers of θ and replacing $r\cos\theta$ and $r\sin\theta$ by x and y, and r^2 by x^2+y^2, the equation of the locus formed by the perimeter may be written down.

When $n=5$, the left side is identical with

$\cos 5\alpha.\ r^5(16\cos^5\theta-20\cos^3\theta+5\cos\theta)+\sin 5\alpha.\ r^5(16\sin^5\theta-20\sin^3\theta+5\sin\theta)$

$=\cos 5\alpha\ (x^5-10x^3y^2+5xy^4)+\sin 5\alpha\ (5x^4y-10x^2y^3+y^5)$,

so that the equation required is

$\cos 5\alpha\ (x^5-\ldots)+\sin 5\alpha\ (5x^4y-\ldots)-5a\ (r^4+20\ a^3r^2-16a^5=0.$

When $n=7$, we get similarly

$\cos 7\alpha(x^7-21x^5y^2+35x^3y^4-7xy^6)+\sin 7\alpha\ (7x^6y-35x^4y^3+21x^2y^5-y^7)$
$+7a\ (r^6r^4r^2-56\ a^3+112\ a^5-64\ a^7=0.$

Question 481.

(R. VYTHYANATHA SWAMY) :—If $x^{n-1}=a_0+a_1(x-1)+a_2(x-1)(x-2)+\ldots a_{n-1}(x-1)(x-2)\ldots(x-n+1)$, shew that

$$(x\,d/dx)^n y=a_0xy_1+a_1x^2y_2+\ldots a_{n-1}x^ny_n.$$

Hence shew that, if $f(x)=A_0+A_1x+A_2x^2+\ldots$,

$A_1 1^n+A_2.2^n+A_3.3^n+\ldots=a_0f'(1)+a_1f''(1)+a_2f'''(1)+\ldots a_{n-1}f^n(1)$

Solution by R. Srinivasan M. A., and V. K. Aravamuthan.

Because $x^n = a_0 x + a_1 x(x-1) + \dots$

$\therefore$ $(xD)^n = a_0\,(xD) + a_1\,xD\,(xD-1) + \dots$

$= a_0\,xD + a_1 x^2 D^2 + \dots + a_{n-1} x^n D^n.$

$\therefore$ $(xD)^n y = a_0\,xy_1 + a_1 x^2 y_2 + \dots + a_{n-1}\,x^n y_n$

Let $y = f\,(x) \equiv A_0 + A_1\,x + A_r x^2 + \dots$

$(xD)^n.\ (A_r x^r) = A_r x^r . r^n.$

$\therefore$ $A_1\,1^n . x + A^2 . 2^n . x^2 + \dots$

$= a_0\,xf'\,(x) + a_1 x^2 f''\,(x) + \dots + a_{n-1}\,f^n\,(x.)$

Put $x=1$; we get

$A_1 . 1^n + A^2 . 2^n + A_3 . 3^n + \dots$

$= a_0\,f'\,(1) + a_1\,f''\,(1) + \dots + a_{n-1}\,f^n\,(1)$

Question 485.

(S. Krishnaswami Aiyangar) :—Normals at P,Q,R,S, on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meet at O. If PQ envelopes $x^{\frac{2}{3}}+y^{\frac{2}{3}}=(ab)^{\frac{1}{3}}$, show that the envelope of RS is a coaxal ellipse.

Solution by R. Srinivasen, M.A., K. Narasinga Rao and N. Sankara Aiyar.

Let PQ and RS be $lx+my=1$, and $l'x+m'y=1$.

$$\text{Then } ll' = -\frac{1}{a^2},\ mm' = -\frac{1}{b^2}.$$

A tangent to $x^{\frac{2}{3}}+y^{\frac{2}{3}}=(ab)^{\frac{1}{3}}$ is

$$xx'^{-\frac{1}{3}} + yy'^{-\frac{1}{3}} = (ab)^{\frac{1}{3}}.$$

Comparing this with $lx+my=1$, we get

$$l=(abx')^{-\frac{1}{3}},\ m=(aby')^{-\frac{1}{3}}$$

$$\therefore \qquad \frac{1}{l^2}+\frac{1}{m^2}=ab,$$

$$i.e. \qquad a^4\,l'^2+b^4 m'^2=ab,$$

Hence $l'x+m'y=1$ touches the ellipse,

$$\frac{bx^2}{a^3}+\frac{ay^2}{b^3}=1,$$

which is coaxal $\qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

Question 488.

(SELECTED) :—Prove that the roots of the determinant equation of the nth degree

$$\begin{vmatrix} x & 1 & 0 & 0 & 0 & \dots & \dots & \dots \\ 1 & x & 1 & 0 & 0 & \dots & \dots & \dots \\ 0 & 1 & x & 1 & 0 & \dots & \dots & \dots \\ \dots & & \dots & & \dots & & \dots & \\ \dots & & \dots & & \dots & & \dots & \end{vmatrix} = 0.$$

are $2\cos\frac{k\pi}{n+1}(k=1, 2, \dots\dots n)$.

Solution by R. Srinivasan, M.A., and V. K. Aravamuthan.

With the usual meanings for Δ_n, $x\Delta_{n-1}$ etc., we get

$$\Delta_n = x\Delta_{n-1} - \Delta_{n-2},$$

[Burnside and Panton, Vol. II, p. 63, Ex. 36.]

or $$\Delta_n - x\Delta_{n-1} + \Delta_{n-2} = 0 \quad \dots \quad \dots \quad \dots \quad (1)$$

Also $$\Delta_1 = x; \; \Delta_2 = x^2 - 1 \quad \dots \quad \dots \quad \dots \quad (2)$$

To solve the difference equation (1), put $x = 2\cos\theta$. Then the equation reduces to

$$\Delta_n - 2\cos\theta\,\Delta_{n-1} + \Delta_{n-2} = 0.$$

The solution of this is $A\cos n\theta + B\sin n\theta$. This contains two arbitrary constants which are to be found from conditions (2).

Now $$2\cos\theta = A\cos\theta + B\sin\theta$$

$$4\cos^2\theta - 1 = A\cos 2\theta + B\sin 2\theta.$$

$\therefore$ $$A = 1,\; B = \cot\theta.$$

$\therefore$ $$\Delta_n = \cot n\theta + \cot\theta\sin n\theta = \sin(n+1)\theta/\sin\theta.$$

The given determinant equation therefore reduces to

$$\sin(n+1)\theta/\sin\theta = 0,$$

which gives $\theta = \frac{k\pi}{n+1}(k=1, 2\dots\dots n)$.

Hence the values of x are given by $2\cos\frac{k\pi}{n+1}\;(k=1, 2, \dots\dots n)$.

Question 491.

(M. T. NARANIENGAR) :—In any triangle prove that a circumconic passing through the ends of a diameter of the maximum inscribed ellipse touches the ellipse.

Solution by N. Sankara Aiyar.

The maximum inscribed ellipse is

$$\Sigma a^2\alpha^2 - 2\,\Sigma ab\alpha\beta = 0.$$

Any conic is

$$\Sigma a^2\alpha^2 - 2\,\Sigma ab\alpha\beta - (l\alpha + m\beta + n\gamma)(p\alpha + q\beta + r\gamma) = 0,$$

where $l\alpha + m\beta + n\gamma = 0$ and $p\alpha + q\beta + \partial\gamma = 0$, are the chords of intersection with the inscribed ellipse.

If this is a circumconic

$$a^2 - lp = 0,\ b^2 - mq = 0,\ c^2 - nq = 0\,;$$

$$\therefore \qquad p = \frac{a^2}{b},\ q = \frac{n^2}{m},\ \gamma = \frac{c^2}{n}.$$

If one of these chords passes through the centre $(1/a,\ 1/b,\ 1/c)$, we find

$$\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 0.$$

The other chord is

$$\frac{a^2\alpha}{l} + \frac{b^2\beta}{m} + \frac{c^2\gamma}{n} + 0.$$

This will touch the inscribed ellipse, if

$$\frac{la}{a^2} + \frac{mb}{b^2} + \frac{nc}{c^2} = 0,$$

i.e.

$$\frac{l}{a} + \frac{m}{b} + \frac{n}{c} = 0,$$

which is the condition that the other chord should pass through the centre.

Hence this chord is the common tangent of the ellipse and the circumconic ; and therefore the circumconic touches the ellipse.

Question 493.

(S. P. SINGARAVELU MODELIAR):—Sum to infinity the following series:

$$(1)\quad \frac{1}{1^2}+\frac{1}{2}\cdot\frac{1}{3^2}+\frac{1\cdot3}{2\cdot4}\,\frac{1}{5^2}+\ldots\ldots\ldots$$

$$(2)\quad \frac{1}{1^3}+\frac{1}{2}\cdot\frac{1}{3^3}+\frac{1\cdot3}{2\cdot4}\,\frac{1}{5^3}+\ldots\ldots\ldots$$

$$(3)\quad \frac{1}{2}\cdot\frac{s_1}{3}+\frac{1}{2\cdot4}\,\frac{s_2}{5}+\frac{1\cdot3}{2\cdot4\cdot6}\,\frac{s_3}{7}+\ldots\ldots\ldots$$

where s_n = the sum of the squares of the reciprocals of the first n odd numbers.

Solution by S. Ramanujan and R. Srinivasan.

Results (1) and (2) have been found to be

$\frac{\pi}{2}$long 2 and $\frac{\pi^3}{48}+\frac{\pi}{4}$ $(\log\ 2)^2$ (*Vide*: Vol. IV pages 33 and 60.)

(3) We know
$$\frac{(\sin^{-1}x)^3}{6}=\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1:3}{2\cdot4}\,\frac{x^5}{5}\left(\frac{1}{1^2}+\frac{1}{3^2}\right)$$
$$+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{x^7}{7}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\right)+\&\text{c.}$$

Hence
$$\frac{1}{2}\frac{x}{3}+\frac{1\cdot1}{2\cdot4}\cdot\frac{x^3}{5}\left(\frac{1}{1^2}+\frac{1}{3^2}\right)+\frac{1\cdot3}{2\cdot4.6}\cdot\frac{x^5}{7}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}\right)+\&\text{c}$$
$$=\int\frac{(\sin^{-1}x)^3}{6x^3}dx$$
$$=-\frac{(\sin^{-1}x)^3}{12x^2}+\int\frac{(\sin^{-1}x)^2}{4x^2\sqrt{1-x^2}}dz$$
$$=-\frac{(\sin^{-1}x)^3}{12x^2}-\frac{\sqrt{1-x^2}}{4x}(\sin^{-1}x)^2+\int\frac{\sin^{-1}x}{2x}dx$$

Hence
$$\frac{1}{2}\frac{1}{3}s_1+\frac{1\cdot}{2\cdot4}\frac{1}{5}s_2+\frac{1\cdot3}{2\cdot4\cdot6}\frac{1}{7}\cdot s_3+\&\text{c}=\int_0^1\frac{(\sin^{-1}x)}{6x^3}\,dx$$
$$=-\frac{\pi^3}{96}+\tfrac{1}{2}\int_0^1\frac{\sin^{-1}x}{x}dx=\frac{\pi\log2}{4}-\frac{\pi^3}{96},$$
since $\int_0^1\frac{\sin^{-1}x}{x}dx=\frac{\pi}{2}\log\ 2$, by (1).

QUESTIONS FOR SOLUTION.

497. (T. P. Trivedi, M.A., L.L.B.) :—Prove that

$$\int_0^\infty \frac{\sin rx}{x(1+x^2)^2}dx=\frac{\pi}{4}\left(2-e^{-r}(r+2)\right),$$

$$\int_0^\infty \frac{\sin^2 rx}{x^2(1+x^2)}dx=\frac{\pi}{4}\left(-1+e^{2-r}+2r\right).$$

498. (R. N. Apte, M.A., F.R.A.S) :—The parabola $y=ax^2+bx+c$ is drawn at random (where a, b, c are real). Show that the probability of its cutting the x—axis in imaginary points is $\left(\frac{31}{72}-\frac{1}{12}\log 2\right)$.

499. (K. J. Sanjana) :—A chord, AKB, of a course is drawn perpendicular to the normal PK at any point P of the curve, and M is its mid-point; prove that, when PK is infinitesimal the ratio of the radius of curvature at P to the radius of curvature at the corresponding point of the evolute is PK : 3 KM. (Suggested by Question 470.)

500. (V. K. Aravamudan) :—Solve the equation

$$\frac{dy}{dx}+y^2+\frac{2y}{x}+x^2-\frac{2}{x^2}=0.$$

501. (K. J. Sanjana, M.A.) :—If $\overset{r}{\underset{n}{S}}\equiv 1^r+2^r+\ldots+n^r$, prove that

$$\overset{r}{\underset{n}{S}}-\binom{r+1}{1}\overset{r}{\underset{n-1}{S}}+\binom{r+1}{2}\overset{r}{\underset{n-1}{S}}\ldots+(-1)^{r+1}\binom{r+1}{r+1}\overset{r}{\underset{n+r-1}{S}}=\underline{|r},$$

where n is any integer greater than $r+1$. When $n=r+1$ or r, prove that the formula is still true if the last or the last two terms are neglected; and find the value of the series when $n=r-1$.

502. (T. P. Trivedi, M.A., L.L.B.) :—Prove that the equation of the sides of a regular polygon of $2n+1$ sides may be written in the form

$$\{ y^{2n+1}-{}^{2n+1}C_2 y^{2n-1}x^2+{}^{2a+1}C_4 y^{2n-3}x^4-\ldots+$$
$$(-1)^n(2n+1)yx^{2n} \} \sin(2n+1)\alpha$$
$$+(-1)^n \{ x^{2n+1}-{}^{2n+1}C_2 x^{2n-1}y^2+{}^{2n+1}C_4 x^{2n-2}h^4\ldots+(-1)^n(2n+1)$$
$$xy^{2n} \} \cos(2n+1)\alpha$$
$$-\left\{ nar^{1-n}-\frac{n(n^2-1^2)}{\underline{|3}}a^3r^{2n-2}+\frac{n(n^2-1^2)(n^2-3^2)}{\underline{|6}}a^3r^{2n-3}\ldots\text{to}\right.$$
$$\left. n+1 \text{ terms} \right\}=0.$$

[Extension of Ques. 479.]

503. (R. N. Apte, M.A., L.L.B.).—A circle is drawn within an ellipse concentric with the ellipse. A tangent to this circle cuts the

ellipse in P,P′. The tangents to the ellipse $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\right)$ at P,P″ meet in T, the normals in N. If p is the perpendicular from the centre on TN; θ is the angle between PP′ and TN; $a^2+\lambda$, $a^2+\mu$ are the squares of the transverse semi-axes of the confocal conics through T; prove that for the different positions PP′

$$\frac{1}{\lambda}+\frac{1}{\gamma}\propto p\sec\theta.$$

504. (R. Srinivasan M.A.) :—Shew that

$$\int_0^{\pi}\theta\sin\theta\cos^{2m}\theta\cos(\alpha\cos\theta)\,d\theta$$

$$=\frac{\pi}{\alpha^{2m+1}}\left\{P\sin\alpha+Q\cos\alpha\right\}$$

where $P=\alpha^{2m}-2m(2m-1)\alpha^{2-2}+2m(2m-1)(2m-2)(2m-3)\alpha^{2m-4}-\ldots$

$$Q=2m\alpha^{2m-1}-2m(2m-1)(2m-2)\alpha^{2m-3}+\ldots\ldots\ldots$$

505. (A. C. L. Wilkinson, M. A., F. R. A. S.) :—A point P moves along a curve and PQ is drawn making an angle with the tangent at P equal to λ times the angle which the tangent at P makes with the tangent at a fixed point O of the curve. If PQ envelops a circle fine the equation of the curve and trace the curves for which $\lambda=1$, $\lambda=\frac{1}{2}$ showing that the latter is a unicursal algebraical curve of the fifth degree whose equation is

$$108x^4a+x^2(y^3+162ay^2+405a^2y-2916a^3)+y(y^2+26ay-81a^2)^2=0,$$

where a is the radius of the given circle and that the former has an infinite number of parabolic asymptotes given by the equation,

$$y^2=4ar\pi(-x+ar\pi),$$

where r is a positive or negative integer.

506. (A. C. L. Wilkinson, M. A., F. R. A. S.) :—Show that it is impossible to describe a plane quadrilateral such that the angle between the diagonals is equal to each of the angles between the two pairs of opposite sides.

507. (S. Ramanujan) :—Solve completely $x^2=y+a$, $y^2=z+a$, $z^2=x+a$; and hence shew that

(*a*) $\sqrt{8-\sqrt{8+\sqrt{8-\&c.}}}=1+2\sqrt{3}\sin 20^\circ.$

(*b*) $\sqrt{11-2\sqrt{11+2\sqrt{11-\&c.}}}=1+4\sin 10^\circ.$

(*c*) $\sqrt{23-2\sqrt{23+2\sqrt{23+2\sqrt{23-2\sqrt{23+\&c.}}}}}$, $=1+4\sqrt{3}\sin 20^\circ.$

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